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Non-Optimal Rates of Ergodic Limits and Approximate Solutions

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This paper is concerned with non-optimal rates of convergence for two processes $\{A_\alpha\}$ and $\{B_\alpha\}$, which satisfy $\|A_\alpha\| = O(1)$, $B_\alpha A \subset AB_\alpha = I - A_\alpha$, $\|AA_\alpha\| = O(e(\alpha))$, where A is a closed operator and $e(\alpha) \to 0$. Under suitable conditions, we describe, in terms of K-functionals, those x (resp. y) for which the non-optimal convergence rate of $\{A_\alpha x\}$ (resp. $\{B_\alpha y\}$) is of the order $O(f(\alpha))$, where f is a function satisfying $e(\alpha) \leqslant f(\alpha) \to 0$. In case that $f(\alpha)/e(\alpha) \to \infty$, the sharpness of the non-optimal rate of $\{A_\alpha x\}$ is equivalent to that A has non-closed range. The result provides a unified approach to dealing with non-optimal rates for many particular mean ergodic theorems and for various methods of solving the equation $Ax = y$. We discuss in particular applications to α-times integrated semigroups, n-times integrated cosine functions, and tensor product semigroups. © 1998 Academic Press

1. INTRODUCTION

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X. Let $\{T(t); t \geqslant 0\} \subset B(X)$ be a uniformly bounded C_0-semigroup with infinitesimal generator A. It is well know (see, e.g., [11, p. 688]) that the Cesáro mean $C(t) := t^{-1} \int_0^t T(s)\, ds$ of $T(\cdot)$ converges strongly on $X_0 = N(A) \oplus \overline{R(A)}$ to the projection P which has range $N(A)$ and null space $\overline{R(A)}$. It was proved by Butzer and Dickmeis [3] that concerning optimal convergence one has

$$\|C(t)\, x - Px\| = O(t^{-1}) \qquad [\text{resp. } o(t^{-1})] \qquad (t \to \infty)$$
$$\Leftrightarrow x \in [D(B_0)]^{\sim}_{X_0} \qquad [\text{resp. } x \in N(A)], \tag{1}$$

where $B_0 = 0 \oplus B_1$ with B_1 the inverse operator of $A_1 := A|_{\overline{R(A)}}$, and $[D(B_0)]^{\sim}_{X_0}$ is the completion of $D(B_0)(= N(A) \oplus R(A_1))$ relative to X_0.

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Concerning the non-optimal convergence rates, one has that for $0<\beta\leqslant 1$

$$\|C(t)\,x-Px\|=O(t^{-\beta})\qquad (t\to\infty)$$
$$\Leftrightarrow K(t^{-1},x,X_0,D(B_0),\|\cdot\|_{B_0})=O(t^{-\beta})\qquad (t\to\infty),$$

the latter being Peetre's K-functional, i.e.,

$$K(t^{-1},x,X_0,D(B_0),\|\cdot\|_{B_0}):=\inf\{\|x-y\|+t^{-1}\|y\|_{B_0};\,y\in D(B_0)\}.$$

In [17] it is proved that if B_0 is unbounded (as observed in [22], this condition is equivalent to that $R(A)$ is not closed), then for each $0<\beta\leqslant 1$ there exists $x_\beta\in X_0$ such that

$$\|C(t)\,x_\beta-Px_\beta\|\begin{cases}=O(t^{-\beta})\\ \neq o(t^{-\beta})\end{cases}\qquad (t\to\infty).\tag{3}$$

Thus, if $R(A)$ is not closed, then for any $0<\alpha<\beta\leqslant 1$ there exists an $x_\gamma\in X_0$ (with $\alpha<\gamma<\beta$) such that $\|C(t)\,x_\gamma-Px_\gamma\|=o(t^{-\alpha})$ but $\neq O(t^{-\beta})$. In particular, we have $O(t^{-1})=[D(B_0)]^{\sim}_{X_0}\subset(\neq)\,o(t^{-\alpha})\subset(\neq)\,o(1)=X_0$. It is known that if $R(A)$ is closed, then not only $O(t^{-1})=N(A)\oplus R(A)=X_0=X$, but also $\|C(t)-P\|=O(t^{-1})$ $(t\to\infty)$.

The aim of this paper is to present an abstract framework for non-optimal rates of convergence for ergodic limits and for approximate solutions of linear functional equations, and to apply the general results to various particular examples. For that purpose we consider the non-optimal convergence of the following processes.

Let $A: D(A)\subset X\to X$ be a closed linear operator, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ satisfying:

(C1) $\|A_\alpha\|\leqslant M$ for all α;

(C2) $R(B_\alpha)\subset D(A)$ and $B_\alpha A\subset AB_\alpha=I-A_\alpha$ for all α;

(C3) $R(A_\alpha)\subset D(A)$ for all α, and $\|AA_\alpha\|=O(e(\alpha))$;

(C4) $B_\alpha^* x^*=\varphi(\alpha)\,x^*$ for all $x^*\in R(A)^\perp$, and $|\varphi(\alpha)|\to\infty$;

(C5) $\|A_\alpha x\|=O(f(\alpha))$ (resp. $o(f(\alpha))$) implies $\|B_\alpha x\|=O(f(\alpha)/e(\alpha))$ (resp. $o(f(\alpha)/e(\alpha))$),

where e and f are positive functions satisfying $0\leqslant e(\alpha)\leqslant f(\alpha)\to 0$.

Note that (C2) implies $A_\alpha A\subset AA_\alpha$ for all α. The functions $e(\alpha)$ and $f(\alpha)$ are to act as estimators of the convergence rates of $\{A_\alpha x\}$ and $\{B_\alpha y\}$, approximating respectively the ergodic limit and the solution of $Ax=y$, in practical applications. The assumptions (C4) and (C5) play key roles in the proof of our theorems and prevail among practical examples.

Strong ergodic theorems, uniform ergodic theorems, and saturation theorems for this general framework have been discussed in [20, 22, 23] respectively, and they subsume many particular results for various systems of operators. For convenience of reference, these general theorems will be quoted in Section 2. The main results, to be stated and proved in Section 3, describe non-optimal rates of convergence of A_α and B_α in terms of the convergence order of K-functionals, and justify the sharpness of non-optimal rates of convergence. Applications to particular examples, such as α-times integrated semigroups, n-times integrated cosine functions, and tensor product semigroups, will then be discussed in Section 4.

2. PRELIMINARY RESULTS

Let P and B_1 be the operators defined respectively by

$$\begin{cases} D(P) := \{x \in X;\ \lim_\alpha A_\alpha x \text{ exists}\}; \\ Px := \lim_\alpha A_\alpha x \qquad \text{for} \quad x \in D(P), \end{cases}$$

and

$$\begin{cases} D(B_1) := \{y \in X;\ \lim_\alpha B_\alpha y \text{ exists}\}; \\ B_1 x := \lim_\alpha B_\alpha y \qquad \text{for} \quad y \in D(B_1). \end{cases}$$

The following strong mean ergodic theorem for the systems $\{A_\alpha\}$ and $\{B_\alpha\}$ is proved in [20, Theorem 1.1, Corollary 1.4, and Remark 1.7].

THEOREM A (Strong Ergodic Theorem). *Under conditions* (C1)–(C4) *the following are true.*

(i) *P is a bounded linear projection with range $R(P) = N(A)$, null space $N(P) = \overline{R(A)}$, and domain $D(P) = N(A) \oplus \overline{R(A)} = \{x \in X;\ \{A_\alpha x\}$ has a weak cluster point$\}$.*

(ii) *B_1 is the inverse operator A_1^{-1} of the restriction $A_1 := A \mid \overline{R(A)}$ of A to $\overline{R(A)}$; it has range $R(B_1) = D(A_1) = D(A) \cap \overline{R(A)}$ and domain $D(B_1) = R(A_1) = A(D(A) \cap \overline{R(A)})$. Moreover, for each $y \in D(B_1)$, $B_1 y$ is the unique solution of the functional equation $Ax = y$ in $\overline{R(A)}$.*

(iii) *$\{A_\alpha\}$ is strongly ergodic, i.e., $D(P) = X$, if and only if $N(A)$ separates $R(A)^\perp$, if and only if $\{A_\alpha x\}$ has a weak cluster point for each $x \in X$. In this case, we have $R(A) = D(B_1) = A(D(A) \cap \overline{R(A)})$. These are true in particular when X is reflexive.*

Let $X_1 := \overline{R(A)}$ and $X_0 := D(P) = N(A) \oplus X_1$. Since the operator $B_1 : D(B_1) \subset X_1 \to X_1$ is closed, its domain $D(B_1)$ $(=R(A_1))$ is a Banach space with respect to the norm $\|x\|_{B_1} := \|x\| + \|B_1 x\|$. The completion of $D(B_1)$ relative to X_1, denoted by $[D(B_1)]^{\sim}_{X_1} (=[R(A_1)]^{\sim}_{X_1})$, is the set of all those $y \in X_1$ for which there exist a sequence $\{y_n\} \subset D(B)$ and a constant $K > 0$ such that $\|y_n\|_B \leqslant K$ for all n and $\|y_n - y\| \to 0$. Let $B_0 : D(B_0) \subset X_0 \to X_0$ be the operator $B_0 := 0 \oplus B_1$. Then its domain $D(B_0)$ $(= N(A) \oplus D(B_1) = N(A) \oplus A(D(A) \cap \overline{R(A)}))$ is a Banach space with norm $\|x\|_{B_0} := \|x\| + \|B_0 x\|$, and $[D(B_0)]^{\sim}_{X_0} = N(A) \oplus [D(B_1)]^{\sim}_{X_1}$.

The following lemma is an immediate consequence of the above definition of the operator B_0.

LEMMA. *Let X_0, A, B_0, and P be as defined previously. Then*

(1) $PB_0 x = 0$ *for* $x \in D(B_0)$, *and* $B_0 P x = 0$ *for* $x \in X_0$.

(2) $AB_0 x = x - Px$ *for* $x \in D(B_0)$, *and* $B_0 A x = x - Px$ *for* $x \in N(A) \oplus (D(A) \cap \overline{R(A)})$.

Concerning optimal convergence, the following theorem from [23] characterizes the Favard (or saturation) classes for the two processes $\{A_\alpha\}$ and $\{B_\alpha\}$.

THEOREM B (Saturation Theorem). *Under conditions* (C1)–(C5) *with* $f(\alpha) = e(\alpha)$, *the following are true.*

(i) *For* $x \in X_0$ *one has* $\|A_\alpha x - Px\| = O(e(\alpha))$ (*resp.* $o(e(\alpha))$) *if and only if* $x \in [D(B_0)]^{\sim}_{X_0}$ (*resp.* $x \in N(A)$).

(ii) *For* $x \in X$ *one has* $\|B_\alpha x\| = O(1)$ (*resp.* $o(1)$) *if and only if* $x \in [D(B_1)]^{\sim}_{X_1}$ (*resp.* $x = 0$).

(iii) *For* $y \in D(B_1) = R(A_1)$ *one has* $\|B_\alpha y - B_1 y\| = O(e(\alpha))$ (*resp.* $o(e(\alpha))$) *if and only if* $y \in A(D(A) \cap [D(B_1)]^{\sim}_{A_1})$ (*resp.* $y = 0$).

The next theorem from [22] will be needed in discussing the sharpness of non-optimal rates. One can see that the original assumption that A is densely defined is unnecessary.

THEOREM C (Uniform Ergodic Theorem). *Under conditions* (C1)–(C3), *we have* $D(P) = X$ *and* $\|A_\alpha - P\| \to 0$ *if and only if* $\|B_\alpha|_{R(A)}\| = O(1)$, *if and only if* B_1 *is bounded and* $\|B_\alpha|_{R(A)} - B_1\| \to 0$, *if and only if* $R(A)$ (*or* $R(A_1)$) *is closed. Moreover, the convergence of these limits has order* $O(e(\alpha))$.

Next, for easier application to Theorem 2 we formulate the following version of the condensation theorem of Davydov [10, Theorem 1].

THEOREM D. *Let* $\{p_\alpha\}$ *be a net of continuous seminorms on a Banach space* X *satisfying the conditions*:

(a) $\overline{\lim}_\alpha \|p_\alpha\| = \infty$, *where* $\|p_\alpha\| := \sup\{p_\alpha(x); x \in X, \|x\| \leqslant 1\}$;

(b) *the set* $\{x \in X; \lim_\alpha p_\alpha(x) = 0\}$ *is dense in* X.

Then there exists an element $x_0 \in X$ *such that* $\sup_\alpha p_\alpha(x_0) \leqslant 1$ *and* $\overline{\lim}_\alpha p_\alpha(x_0) = 1$.

3. MAIN RESULTS

In this section we prove two main theorems of this paper; the first one shows that the non-optimal convergence rates of ergodic limits and approximate solutions are the same as those of the corresponding K-functionals; the second one shows the sharpness of the non-optimal rates in the case that $R(A)$ is not closed.

THEOREM 1. *Under conditions* (C1)–(C5) *the following statements hold.*

(i) *For* $x \in X_0 = N(A) \oplus \overline{R(A)}$, *one has* $\|A_\alpha x - Px\| = O(f(\alpha))$ *if and only if* $K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(f(\alpha))$.

(ii) *For* $x \in \overline{R(A)}$, *one has* $\|A_\alpha x\| = O(f(\alpha))$ *if and only if* $K(e(\alpha), x, X_1, D(B_1), \|\cdot\|_{B_1}) = O(f(\alpha))$.

(iii) *For* $y \in D(B_1) = R(A_1)$ *one has* $\|B_\alpha y - B_1 y\| = O(f(\alpha))$ *if and only if* $K(e(\alpha), B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(f(\alpha))$.

Since K-functionals are saturated (see [2, p. 15]), i.e., $K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(e(\alpha))$ if and only if $x \in [D(B_0)]^{\sim}_{X_0}$, and $K(e(\alpha), y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(e(\alpha))$ if and only if $y \in [D(B_1)]^{\sim}_{X_1}$, Theorem B also follows from Theorem 1.

Proof of Theorem 1. First we show the sufficiency part of (i). Using (C1), (C3), and the Lemma in Section 2 we have for any fixed $x \in X_0$ and arbitrary $y \in D(B_0)$

$$\begin{aligned}
\|A_\alpha x - Px\| &\leqslant \|A_\alpha x - A_\alpha y\| + \|A_\alpha y - Py\| + \|Py - Px\| \\
&\leqslant (\|A_\alpha\| + \|P\|)\, \|x - y\| + \|A_\alpha(y - Py)\| \\
&\leqslant 2M\, \|x - y\| + \|A_\alpha A B_0 y\| \\
&\leqslant 2M\, \|x - y\| + ke(\alpha)\, \|B_0 y\| \\
&\leqslant (2M + k)[\, \|x - y\| + e(\alpha)\, \|y\|_{B_0}].
\end{aligned}$$

Hence $\|A_\alpha x - Px\| \leqslant (2M + k)\, K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(f(\alpha))$.

The necessity. For $x \in X_0$ let $x_\alpha = x + Px - A_\alpha x$. Using (C2) and the fact that $R(P) = N(A)$ we can write

$$x_\alpha = Px + AB_\alpha x = Px + AB_\alpha x - B_\alpha APx = Px + AB_\alpha(x - Px).$$

Because (C2) implies $B_\alpha \overline{R(A)} \subset D(A) \cap \overline{R(A)}$ and $x - Px$ lies in $\overline{R(A)}$, we have $B_\alpha(x - Px) \in D(A) \cap \overline{R(A)}$ and $x_\alpha \in N(A) \oplus A(D(A) \cap \overline{R(A)} = D(B_0)$. Then, applying B_0 and using the Lemma in Section 2 we obtain $B_0 x_\alpha = B_0 Px + B_0 AB_\alpha(x - Px) = B_\alpha(x - Px)$. Since $\|A_\alpha(x - Px)\| = \|A_\alpha x - Px\| = O(f(\alpha))$, (C5) implies that $\|B_0 x_\alpha\| = \|B_\alpha(x - Px)\| = O(f(\alpha)/e(\alpha))$. Hence

$$\begin{aligned}
&K(e(\alpha), x, X_0, D(B_0), \|\cdot\|_{B_0}) \\
&\quad \leqslant \|x - x_\alpha\| + e(\alpha)\, \|x_\alpha\|_{B_0} \\
&\quad \leqslant \|A_\alpha x - Px\| + e(\alpha)[\, \|x + Px - A_\alpha x\| + \|B_0 x_\alpha\|\,] \\
&\quad \leqslant O(f(\alpha)) + e(\alpha)\left[(1 + 2M)\, \|x\| + O\left(\frac{f(\alpha)}{e(\alpha)}\right)\right] = O(f(\alpha)).
\end{aligned}$$

Part (ii) can be proved by slightly modifying the above proof of (i), and (iii) follows from (ii) because we have

$$B_\alpha y - B_1 y = B_\alpha AB_1 y - B_1 y = (B_\alpha A - I)\, B_1 y = -A_\alpha B_1 y$$

for $y \in D(B_1)$.

It follows from (i) of Theorem B that $\|A_\alpha y\| = O(e(\alpha))$ but $\neq o(e(\alpha))$ for every nonzero element y of $[D(B_1)]^{\sim}_{X_1} = [A(D(A) \cap \overline{R(A)})]^{\sim}_{X_1}$. Hence, when $A \neq 0$, $\|A_\alpha y\| = O(e(\alpha))$ is sharp everywhere on $[D(B_1)]^{\sim}_{X_1} \backslash \{0\}$. The following theorem characterizes the sharpness of $\|A_\alpha y\| = O(f(\alpha))$ for f satisfying $f(\alpha)/e(\alpha) \to \infty$.

THEOREM 2. *Suppose that A, $\{A_\alpha\}$, and $\{B_\alpha\}$ satisfy conditions* (C1)–(C5), *with $f(\alpha)/e(\alpha) \to \infty$. Then $R(A)$ is not closed if and only if there exists an element $y_f \in X_1 = \overline{R(A)}$ such that*

$$\|A_\alpha y_f\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)). \end{cases} \tag{4}$$

Proof. The sufficiency. If $R(A)$ is closed, then by Theorem C we have $D(P) = X$ and $\|A_\alpha - P\| = O(e(\alpha))$, so that $\|A_\alpha - P\| = o(f(\alpha))$. Hence there does not exist any y_f in X_1 that satisfies (4). To reason in another way, we see from the closedness of $R(A)$ and $D(P) = X$ that $X_1 = R(A) = D(B_1) = [D(B_1)]^{\sim}_{X_1}$ (Theorem A(iii)) so that $\|A_\alpha y\| = O(e(\alpha))$ (Theorem B(i)) and hence $\|A_\alpha y\| = o(f(\alpha))$ for all $y \in X_1$. This leads to the same conclusion that no element in X_1 satisfies (4) when $R(A)$ is closed.

The necessity. Consider the seminorm p_α on X_1 defined by $p_\alpha(x) = \|A_\alpha x\|/f(\alpha)$, $x \in X_1$. We have $\|p_\alpha\| = \|A_\alpha|_{X_1}\|/f(\alpha) \leqslant \|A_\alpha\|/f(\alpha)$. Since $R(A)$ is not closed, it follows from Theorem C that $\overline{\lim}_\alpha \|A_\alpha - P\| > 0$, so that $\overline{\lim}_\alpha \|A_\alpha|_{X_1}\| > 0$. Hence, with X replaced by X_1, condition (a) of Theorem D is satisfied. To show (b), we see from (i) of Theorem B and the assumption of the theorem that $p_\alpha(x) = O(e(\alpha))/f(\alpha) = o(1)$ for all $x \in D(B_1)$. Since application of Theorem A to the triplet $\{A_1, \{A_\alpha|_{X_1}\}, \{B_\alpha|_{X_1}\}\}$ implies that $\overline{D(B_1)} = \overline{R(A_1)} = N(P|_{X_1}) = X_1$, condition (b) of Theorem D is verified. It follows that there exists $y_f \in X_1$ such that $\sup p_\alpha(y_f) \leqslant 1$ and $\overline{\lim}_\alpha p_\alpha(y_f) = 1$. That means that y_f satisfies (4).

Remarks. (i) $R(A)$ is not closed if and only if there is a $y_f \in X_1 = \overline{R(A)}$ such that for any $x \in N(A)$ the element $x_f := x + y_f$ satisfies

$$\|A_\alpha x_f - Px_f\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)). \end{cases} \tag{5}$$

(ii) When $R(A)$ is closed, Theorem C with theorem A(iii) also shows that $D(B_1) = R(A)$, B_1 is bounded, and $\|B_\alpha|_{D(B_1)} - B_1\| = O(e(\alpha))$, so that $\|B_\alpha|_{D(B_1)} - B_1\| = o(f(\alpha))$. Hence, in this case, there does not exist any z_f in $D(B_1)$ satisfying

$$\|B_\alpha z_f - B_1 z_f\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)). \end{cases} \tag{6}$$

When $R(A)$ is not closed, it is unknown whether or not there exists an element $z_f \in D(B_1)$ such that (6) holds.

(iii) Condition (C4) is needed to show that the existence of the limit $\lim_\alpha B_\alpha y$ for an element $y \in X$ implies $y \in \overline{R(A)}$ (cf. [20, Theorem 1.3]). If we require in the definition of the operator B_1 that $D(B_1) := \{y \in \overline{R(A)};\ \lim_\alpha B_\alpha y \text{ exists}\}$, then all the general results are still true without condition (C4).

4. APPLICATIONS

By applying (i) and (ii) of Theorem 1 to discrete semigroups and C_0-semigroups, one can easily deduce Butzer and Westphal's result in [7, 8], and Butzer and Dickmeis' result in [3], i.e., formulas (1) and (2), and by applying Theorem 2 one can deduce the result of Nasri-Roudsari, Nessel, and Zeler [17], i.e., formula (3). One can also apply Theorems 1 and 2 to deduce the particular result in [9] for resolvent families. These applications can be found in [26].

In this section we shall demonstrate applications to the Abelian ergodic theorem with rates, α-times integrated semigroups, n-times integrated cosine functions, and tensor product semigroups.

4.1. *Abelian Ergodic Theorem with Rates.* Let A be a closed operator such that $0 \in \overline{\rho(A)}$ and such that $\|\lambda(\lambda - A)^{-1}\| = O(1)$ ($\lambda \to 0$, $\lambda \in \rho(A)$). Set $A_\lambda := \lambda(\lambda - A)^{-1}$ and $B_\lambda := -(\lambda - A)^{-1}$, $\lambda \in \rho(A)$. Clearly $\{A_\lambda\}$ and $\{B_\lambda\}$ satisfy conditions (C1)–(C5), with $e(\lambda) = |\lambda|$, $f(\lambda) = |\lambda|^\beta$, $0 < \beta \leqslant 1$, and $\varphi(\lambda) = \lambda^{-1} \to \infty$ as $\lambda \to 0$ (cf. [20, 23]). Then the following known result (see [3, 6]) follows from Theorems 1 and 2 immediately.

THEOREM 3. *Let A be a closed operator such that $0 \in \overline{\rho(A)}$ and $\|\lambda(\lambda - A)^{-1}\| = O(1)$ $(\lambda \to 0)$. Then the following are true for $0 < \beta \leqslant 1$:*

(i) *For $x \in X_0$, one has $\|\lambda(\lambda - A)^{-1} x - Px\| = O(|\lambda|^\beta)$ $(\lambda \to 0)$ if and only if $K(|\lambda|, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(|\lambda|^\beta)$ $(\lambda \to 0)$.*

(ii) *For $y \in D(B_1) = R(A_1)$, one has $\|(A - \lambda)^{-1} y - B_1 y\| = O(|\lambda|^\beta)$ $(\lambda \to 0)$ if and only if $K(|\lambda|, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(|\lambda|^\beta)$ $(\lambda \to 0)$.*

(iii) *$R(A)$ is not closed if and only if for each (some) $0 < \beta < 1$ there exists an element $y_B \in \overline{R(A)}$ such that*

$$\|\lambda(\lambda - A)^{-1} y_\beta\| \begin{cases} = O(|\lambda|^\beta) \\ \neq o(|\lambda|^\beta) \end{cases} \qquad (\lambda \to 0).$$

4.2. *α-Times Integrated Semigroups.* For a positive number α a family $\{T(t); t \geqslant 0\}$ in $B(X)$ is called an *α-times integrated semigroup* if

(S1) $T(\cdot)$ is strongly continuous on $[0, \infty)$ and $T(0) = 0$;

(S2) $T(t)\,T(s) = 1/\Gamma(\alpha)(\int_0^{t+s} - \int_0^t - \int_0^s)(t + s - r)^{\alpha - 1}\,T(r)\,dr$ for $s, t \geqslant 0$.

A C_0-semigroup is called a 0-times integrated semigroup.

N-times integrated semigroups were introduced in [1], and fractionally integrated semigroups have been studied in [14–16].

If $T(\cdot)$ is *nondegenerate* in the sense that $x = 0$ whenever $T(t)\,x = 0$ for all $t > 0$, then there exists uniquely a closed (but not necessarily densely defined) operator A such that $x \in D(A)$ and $Ax = y$ if and only if $T(t)\,x = \int_0^t T(s)\,y\,ds + (t^\alpha/\Gamma(\alpha + 1))\,x$ for all $t \geqslant 0$. This operator A is called the *generator* of $T(\cdot)$. It satisfies $T(t)\,D(A) \subset D(A)$ and $T(t)\,A \subset AT(t)$, and

$$\int_0^t T(x)\,x\,ds \in D(A) \qquad \text{and}$$

$$A \int_0^t T(s)\,x\,ds = T(t)\,x - \frac{t^\alpha}{\Gamma(\alpha + 1)}\,x, \qquad x \in X, \quad t \geqslant 0. \tag{7}$$

If $T(\cdot)$ is *exponentially bounded*, i.e., there are $M \geqslant 0$ and $w \in R$ such that $\|T(t)\| \leqslant Me^{wt}$ for all $t \geqslant 0$, then one has

$$(w, \infty) \subset \rho(A) \qquad \text{and} \qquad (\lambda - A)^{-1} x = \int_0^\infty \lambda^\alpha e^{-\lambda t} T(t)\, x\, dt \tag{8}$$

for $x \in X$ and $\lambda > w$.

Let A_t and B_t be operators defined respectively by

$$A_t d := \Gamma(\alpha+2)\, t^{-\alpha-1} \int_0^t T(s)\, x\, ds$$

and

$$B_t x := -\Gamma(\alpha+2)\, t^{-\alpha-1} \int_0^t \int_0^s T(u)\, x\, du\, ds$$

for $x \in X$ and $t > 0$. In [21–23] we have considered strong convergence, uniform convergence, and a saturation theorem for A_t and B_t as $t \to \infty$ (for the case that α is a natural number). At present, their non-optimal rates of convergence are to be deduced from Theorems 1 and 2.

Assume that $T(\cdot)$ satisfies $\|T(t)\| = O(t^\alpha)$ $(t \to \infty)$. Then by (7) and the fact that A is closed, we can easily see that $\{A_t\}$ and $\{B_t\}$ satisfy conditions (C1)–(C4), with $e(t) = t^{-1}$ and $\varphi(t) = -t/(\alpha+2)$. One can also easily verify that $\|A_t y\| = O(t^{-\beta})$ (resp. $o(t^{-\beta})$) implies $\|B_t y\| = O(t^{-\beta+1})$ (resp. $o(t^{-\beta+1})$). That is, (C5), with $f(t) = t^{-\beta}$, $0 < \beta \leqslant 1$, is satisfied. Hence the theorems in Section 3 can be employed to obtain non-optimal rates for Cesáro ergodic limits. On the other hand, the assumption $\|T(t)\| = O(t^\alpha)$ $(t \to \infty)$ also implies that $(0, \infty) \subset \rho(A)$ and $\|\lambda(\lambda - A)^{-1}\| = O(1)$ $(\lambda \to 0^+)$ so that Theorem 3 applies. Therefore we obtain the next theorem.

THEOREM 4. *Let $\{T(t); t \geqslant 0\}$ be a nondegenerate α-times integrated semigroup with generator A, and suppose $\|T(t)\| \leqslant Mt^\alpha$ for all $t \geqslant 0$.*

(i) *The mapping $P : x \to \lim_{t\to\infty} \Gamma(\alpha+2)\, t^{-\alpha-1} \int_0^t T(x)\, x\, ds$ is a bounded linear projection with $R(P) = N(A)$, $N(P) = \overline{R(A)}$, and $D(P) = N(A) \oplus \overline{R(A)}$. For $0 < \beta \leqslant 1$ and $x \in D(P)$, we have $\|\Gamma(\alpha+2)\, t^{-\alpha-1} \int_0^t T(s)\, x\, ds - Px\| = O(t^{-\beta})$ $(t \to \infty)$ if and only if $\|\lambda(\lambda - A)^{-1} x - Px\| = O(\lambda^\beta)$ $(\lambda \to 0^+)$, if and only if $K(\lambda, x, X_0, D(B_0), \|\cdot\|_{B_0}) = O(\lambda^\beta)$ $(\lambda \to 0^+)$. Moreover, in case $\beta = 1$, these conditions are equivalent to $x \in [D(B_0)]^{\sim}_{X_0}$.*

(ii) *The mapping $B_1 : y \to -\lim_{t\to\infty} \Gamma(\alpha+2)\, t^{-\alpha-1} \int_0^t \int_0^s T(u)\, x\, du\, ds$ is the inverse operator A_1^{-1} of the restriction $A_1 := A \mid \overline{R(A)}$ of A to $\overline{R(A)}$;*

it has range $R(B_1)=D(A)\cap\overline{R(A)}$, *and domain* $D(B_1)=A(D(A)\cap\overline{R(A)})$. *For each* $y\in A(D(A)\cap\overline{R(A)})$, $B_1 y$ *is the unique solution of the functional equation* $Ax=y$ *in* $\overline{R(A)}$. *For* $0<\beta\leqslant 1$ *we have* $\|\Gamma(\alpha+2)\,t^{-\alpha-1}\times\int_0^t\int_0^s T(u)\,y\,du\,ds+A_1^{-1}y\|=O(t^{-\beta})$ $(t\to\infty)$ *if and only if* $\|(\lambda-A)^{-1}\,y+A_1^{-1}y\|=O(\lambda^{\beta})$ $(\lambda\to 0^+)$, *if and only if* $K(\lambda, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1})=O(\lambda^{\beta})$ $(\lambda\to 0^+)$. *Moreover, in case* $\beta=1$, *these conditions are equivalent to* $y\in[D(B_1)]^{\sim}_{X_1}$.

(iii) $R(A)$ *is not closed if and only if for every (some)* $0<\beta<1$ *there are* $y_\beta, y'_\beta\in\overline{R(A)}$ *such that*

$$\|\Gamma(\alpha+2)\,t^{-\alpha-1}\int_0^t T(s)\,y_\beta\,ds\|\begin{cases}=O(t^{-\beta})\\ \neq o(t^{-\beta})\end{cases}\qquad(t\to\infty),$$

$$\|\lambda(\lambda-A)^{-1}\,y'_\beta\|\begin{cases}=O(\lambda^{\beta})\\ \neq o(\lambda^{\beta})\end{cases}\qquad(\lambda\to 0^+).$$

Remarks. When $\alpha=0$, (i) reduces to Butzer and Dickmeis' result [3] on C_0-semigroups, and (iii) becomes the result in [17]. When $\alpha>0$ is a natural number, the assertions for the case $\beta=1$ in (i) and (ii) were proved in [23, Theorem 5].

4.3. *N-Times Integrated Cosine Operator Functions.* A strongly continuous family $\{C(t); t\geqslant 0\}$ of bounded linear operators on X is called an *n-times integrated cosine function* $(n\geqslant 1)$ if $C(0)=0$ and

$$\begin{aligned}2C(t)\,C(s)\,x=\frac{1}{(n-1)!}\Big\{&(-1)^n\int_0^{|s-t|}(|s-t|-u)^{n-1}\,C(u)\,x\,du\\&+\left[\int_0^{s+t}-\int_0^t-\int_0^s\right](t+s-u)^{n-1}\,C(u)\,x\,du\\&+\int_0^t(s-t+u)^{n-1}\,C(u)\,x\,du+\int_0^s(t-s+u)^{n-1}\,C(u)\,x\,du\Big\}\end{aligned}$$

for all $x\in X$ and $s, t>0$. It is called a (0-times integrated) cosine function (see [27]) if

$$C(0)=I\qquad\text{and}\qquad 2C(t)\,C(s)=C(t+s)+C(t-s)\qquad\text{for}\quad t\geqslant s\geqslant 0.$$

$C(\cdot)$ is said to be *nondegenerate* if $C(t)\,x=0$ for all $t>0$ implies $x=0$. In this case, there exists a closed operator A which is uniquely defined as

$$x\in D(A)\qquad\text{and}$$

$$Ax=y\Leftrightarrow C(t)\,x-\frac{t^n}{n!}\,x=\int_0^t(t-u)\,C(u)\,y\,du\qquad\text{for}\quad t\geqslant 0.$$

This operator A is called the *generator* of $C(\cdot)$. It is known that $C(t)\,D(A) \subset D(A)$, $C(t)\,A \subset AC(t)$, and $\int_0^t (t-u)\,C(u)\,x\,du \in D(A)$ and

$$A\int_0^t (t-u)\,C(u)\,x\,du = C(t)\,x - \frac{t^n}{n!}\,x \qquad \text{for} \quad x \in X \quad \text{and} \quad t \geqslant 0. \tag{9}$$

In case that $C(\cdot)$ is exponentially bounded, i.e., $\|C(t)\| \leqslant Me^{wt}$, $t \geqslant 0$, one has

$$(w^2, \infty) \subset \rho(A) \qquad \text{and} \qquad (\lambda^2 - A)^{-1}\,x = \int_0^\infty \lambda^{n-1} e^{-\lambda t} C(t)\,x\,dt \tag{10}$$

for $x \in X$ and $\lambda > w$. For properties of n-times integrated cosine functions see [24]. For 0-times integrated cosine functions see [12, 27].

Let operators A_t and B_t be defined respectively by

$$A_t := \frac{(n+2)!}{t^{n+2}} \int_0^t \int_0^s C(u)\,du\,ds$$

and

$$B_t := -\frac{(n+2)!}{t^{n+2}} \int_0^t \int_0^s \int_0^u \int_0^v C(w)\,dw\,dv\,du\,ds.$$

Assume that $C(\cdot)$ satisfies $\|C(t)\| = O(t^n)$ $(t \to \infty)$. Then by (9) and the fact that A is closed, we can easily see that $\{A_t\}$ and $\{B_t\}$ satisfy conditions (C1)–(C4), with $e(t) = t^{-2}$ and $\varphi(t) = -t^2/(n+3)(n+4)$. One can also easily verify that $\|A_t y\| = O(t^{-2\beta})$ (resp. $o(t^{-2\beta})$) implies $\|B_t y\| = O(t^{-2\beta+2})$ (resp. $o(t^{-2\beta+2})$). That is, (C5), with $f(t) = t^{-2\beta}$, $0 < \beta \leqslant 1$, is satisfied. Hence the theorems in Section 3 can be employed to obtain non-optimal rates for Cesáro ergodic limits. On the other hand, the assumption $\|C(t)\| = O(t^n)$ $(t \to \infty)$ also implies that $(0, \infty) \subset \rho(A)$ and $\|\lambda(\lambda - A)^{-1}\| = O(1)$ $(\lambda \to 0^+)$ so that Theorem 3 applies. Therefore we obtain the next theorem.

THEOREM 5. *Let $\{C(t); t \geqslant 0\}$ be a nondegenerate n-times integrated cosine function with generator A, and suppose $\|C(t)\| \leqslant Mt^n$ for all $t \geqslant 0$. We have*:

(i) *The mapping $P : x \to \lim_{t\to\infty} A_t x$ is a bounded linear projection with $R(P) = N(A)$, $N(P) = \overline{R(A)}$, and $D(P) = N(A) \oplus \overline{R(A)}$. For $0 < \beta \leqslant 1$ and $x \in N(A) \oplus R(A)$, we have $\|A_t x - Px\| = O(t^{-2\beta})$ $(t \to \infty)$ if and only if $\|\lambda(\lambda - A)^{-1}\,x - Px\| = O(\lambda^\beta)$ $(\lambda \to 0^+)$, if and only if $K(\lambda, x, X_0, D(B_0)$,*

$\|\cdot\|_{B_0}) = O(\lambda^\beta)$ $(\lambda \to 0^+)$. *Moreover, in case* $\beta = 1$, *these conditions are equivalent to* $x \in [D(B_0)]^{\sim}_{X_0}$.

(ii) *The mapping* $B_1 : y \to \lim_{t\to\infty} B_t x$ *is the inverse operator* A_1^{-1} *of the restriction* $A_1 := A\,|\,\overline{R(A)}$ *of* A *to* $\overline{R(A)}$; *it has range* $R(B_1) = D(A) \cap \overline{R(A)}$, *and domain* $D(B_1) = A(D(A) \cap \overline{R(A)})$. *For each* $y \in A(D(A) \cap \overline{R(A)})$, $B_1 y$ *is the unique solution of the functional equation* $Ax = y$ *in* $\overline{R(A)}$. *For* $0 < \beta \leqslant 1$ *and* $y \in A(D(A) \cap \overline{R(A)})$ *on has* $\|B_t y - A_1^{-1} y\| = O(t^{-2\beta})$ $(t \to \infty)$ *if and only if* $\|(\lambda - A)^{-1} y + A_1^{-1} y\| = O(\lambda^\beta)$ $(\lambda \to 0^+)$, *if and only if* $K(\lambda, B_1 y, X_1, D(B_1), \|\cdot\|_{B_1}) = O(\lambda^\beta)$ $(\lambda \to 0^+)$. *Moreover, in case* $\beta = 1$, *these conditions are equivalent to* $y \in [D(B_1)]^{\sim}_{X_1}$.

(iii) $R(A)$ *is not closed if and only if for every* (*some*) $0 < \beta < 1$ *there are* y_β, $y'_\beta \in \overline{R(A)}$ *such that*

$$\|A_t y_\beta\| \begin{cases} = O(t^{-2\beta}) \\ \neq o(t^{-2\beta}) \end{cases} \qquad (t \to \infty)$$

and

$$\|\lambda(\lambda - A)^{-1}\, y'_\beta\| \begin{cases} = O(\lambda^\beta) \\ \neq o(\lambda^\beta) \end{cases} \qquad (\lambda \to 0^+).$$

Remarks. The strong mean ergodic theorem for $(C, 2)$-means of $C(\cdot)$ for the case $n = 0$ and the part in (i) about non-optimal rates are proved in [18] (see also [20] for the strong ergodic theorem), and the assertions in (i) and (ii) about optimal rates are proved in [23]. Parts (i) and (iii) for the case $n = 0$ can also be found in [4–6].

4.4. *Tensor Product Semigroups.* For $i = 1, 2$, let X_i be a Banach space and $\{T_i(t);\, t \geqslant 0\} \subset B(X_i)$ be a (C_0)-semigroup with the infinitesimal generator A_i. Suppose $\|T_i(t)\| \leqslant M_i e^{w_i t}$, $t \geqslant 0$, $i = 1, 2$. The family $\{\mathbf{S}(t);\, t \geqslant 0\}$ of operators on $B(X_2, X_1)$, defined by $\mathbf{S}(t)\, E = T_1(t)\, E T_2(t) (E \in B(X_2, X_1))$, is a semigroup in the algebra $B(B(X_2, X_1))$, and is called the tensor product semigroup of $T_1(\cdot)$ and $T_2(\cdot)$. The generator Δ of $\mathbf{S}(\cdot)$, defined by the strong operator limit $\Delta E := \text{so-}\lim_{t\to 0^+} t^{-1}(\mathbf{S}(t)\, E - E)$, is closed relative to the weak operator topology and densely defined relative to the strong operator topology (see [19, Proposition 3.3]); it is precisely the operator which has as its domain the set of all those $E \in B(X_2, X_1)$ for which $ED(A_2) \subset D(A_1)$ and $A_1 E + EA_2$ is bounded on $D(A_2)$, and sends each such E to $\overline{A_1 E + EA_2}$. For $\lambda > w_1 + w_2$, $\lambda - \Delta$ is invertible and

$$(\lambda - \Delta)^{-1}\, Ex = \int_0^\infty e^{-\lambda t} (\mathbf{S}(t)\, E)\, x\, dt \qquad (E \in B(X_2, X_1),\, x \in X_2).$$

If $w_1 + w_2 \leqslant 0$, then $(0, \infty) \subset \rho(\Delta)$ and $\|\lambda(\lambda - \Delta)^{-1}\| \leqslant M_1 M_2$ for all $\lambda > 0$, so that Theorem 3 can be applied to Δ.

For $t>0$ define the operators A_t and B_t by

$$(A_t E)\, x := t^{-1} \int_0^t T_1(s)\, E T_2(s)\, x\, ds$$

and

$$(B_t E)\, x := -t^{-1} \int_0^t \int_0^s T_1(u)\, E T_2(u)\, x\, du\, ds$$

for $E \in B(X_2, X_1)$ and $x \in X_2$. If $w_1 + w_2 \leqslant 0$, then $\{A_t\}$, $\{B_t\}$ satisfy conditions (C1)–(C5), with $e(t) = t^{-1}$, $f(t) = t^{-\beta}$, $0 < \beta \leqslant 1$, and $\varphi(t) = -\frac{1}{2} t$. Hence Theorems 1 and 2 yield the following theorem.

THEOREM 6. *Suppose that* $w_1 + w_2 \leqslant 1$, *and let* $\Pi : N(\Delta) \oplus \overline{R(\Delta)} \to N(\Delta)$ *be the projection with* $R(\Pi) = N(\Delta)$ *and* $N(\Pi) = \overline{R(\Delta)}$, *where the overbar denotes the uniform operator closure. We have*:

(i) *For* $0 < \beta \leqslant 1$ *and* $E \in D(\Pi) = N(\Delta) \oplus \overline{R(\Delta)}$, *one has* $\| t^{-1} \int_0^t T_1(s) E T_2(s)\, ds - \Pi E \| = O(t^{-\beta})$ $(t \to \infty)$ *if and only if* $\| \lambda(\lambda - \Delta)^{-1} E - \Pi E \| = O(\lambda^\beta)$ $(\lambda \to 0^+)$, *if and only if* $K(\lambda, D(\Pi), N(\Delta) \oplus \Delta(D(\Delta) \cap \overline{R(\Delta)})) = O(\lambda^\beta)$ $(\lambda \to 0^+)$. *Moreover, in case* $\beta = 1$, *these conditions are equivalent to that* $E \in N(\Delta) \oplus [\Delta(D(\Delta) \cap \overline{R(\Delta)})]^{\sim}_{\overline{R(\Delta)}}$.

(ii) *For* $0 < \beta \leqslant 1$ *and* $F \in \Delta(D(\Delta) \cap \overline{R(\Delta)})$ *one has* $\| t^{-1} \int_0^t \int_0^s T_1(u) F T_2(u)\, du\, ds + (\Delta|_{\overline{R(\Delta)}})^{-1} F \| = O(t^{-\beta})$ $(t \to \infty)$ *if and only if* $\| (\lambda - \Delta)^{-1} F + (\Delta|_{\overline{R(\Delta)}})^{-1} F \| = O(\lambda^\beta)$ $(\lambda \to 0^+)$, *if and only if* $K(\lambda, \overline{R(\Delta)}, \Delta(D(\Delta) \cap \overline{R(\Delta)})) = O(\lambda^\beta)$ $(\lambda \to 0^+)$. *Moreover, in case* $\beta = 1$, *these conditions are equivalent to* $F \in [\Delta(D(\Delta) \cap \overline{R(\Delta)})]^{\sim}_{\overline{R(\Delta)}}$.

(iii) $R(\Delta)$ *is not closed if and only if for every (some)* $0 < \beta < 1$ *there are* $F_\beta, F'_\beta \in \overline{R(\Delta)}$ *such that*

$$\left\| t^{-1} \int_0^t T_1(s)\, F_\beta T_2(s)\, ds \right\| \begin{cases} = O(t^{-\beta}) \\ \neq (t^{-\beta}) \end{cases} \qquad (t \to \infty)$$

and

$$\| \lambda(\lambda - \Delta)^{-1} F'_\beta \| \begin{cases} = O(\lambda^\beta) \\ \neq o(\lambda^\beta) \end{cases} \qquad (\lambda \to 0^+).$$

Remarks. Ergodic limits of $\mathbf{S}(\cdot)$ and approximate solutions of the operator equation $A_1 E + E A_2 = F$ were studied in [19, 20], and their optimal rates of convergence were discussed in [23].

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